Solutions for the Bell-Shaped Lift Distribution

Lifting-Line Solution for the Bell-Shaped Lift Distribution

The aerodynamic theory of the lifting line was first solved by Multhopp[1]. This solution uses a spanwise distribution of the control points accordingly a sinus-function and is applicable to unswept wings. For the bell-shaped lift distribution $c_l t \propto \sin^3\Theta$ a direct solution of the lifting line formula can be derived and is described here.
The integral for total lift c_L can be written as (Truckenbrodt[2]):

$$c_L = \frac{1}{A} \cdot \int_{-s}^{s} c_l \cdot t \cdot dx \qquad \text{(Tru. 5.42)}$$

Tru.

For a symetrical lift distribution this formula can be transformed:

$$c_L = \frac{2 \cdot s}{A} \cdot \int_{1}^{0} c_l \cdot t \cdot d\overline{x} \quad \text{with} \quad c_l \cdot t = [c_l \cdot t]_0 \cdot h(\overline{x}); \quad \overline{x} = x/s \qquad \text{(Equ. 00-01)}$$

If the spanwise control points are aranged along a "central-angle" Θ similar to the Multhopp-procedure, the following terms can be written:

local lift load $h(\overline{x}) = \sin^3\Theta$ and spanwise station $\overline{x} = \cos\Theta$ (Mul. 07)

Mul.

This leads to the final integration and its solution:

$$c_L = \frac{2 \cdot s}{A} \cdot [c_l \cdot t]_0 \int_{0}^{\pi/2} \sin^4\Theta \cdot d\Theta \quad \text{with} \quad d\overline{x} = -\sin\Theta \cdot d\Theta \qquad \text{(Equ. 00-02)}$$

$$c_L = \frac{3\pi \cdot b}{16 \cdot A} \cdot [c_l \cdot t]_0 \qquad \text{(Equ. 00-03)}$$

The total required geometrical twist α_{geo} is, accordingly to Truckenbrodt, the sum of effective α_{eff} and induced angle of attack α_i. Angles are calculated in [rad.]:

$$\alpha_{geo} = \alpha_{eff} + \alpha_i \qquad \text{(Tru 7.64)}$$

Tru.

The local effective angle of attack can be found with the term:

$$[c_l \cdot t] = c_{l\alpha\infty} \cdot t \cdot \alpha_{eff} \qquad \text{(Tru. 7.11 \& 7.28)}$$

Tru.

This term is now used in Equation (Equ. 00-01 b):

1 Die Berechnung der Auftriebsverteilung von Tragflügeln, H. MULTHOPP, 1938
2 Aerodynamik des Flugzeuges, SCHLICHTING/ TRUCKENBRODT, Springer, 2. Aufl. 1976

Solutions for the Bell-Shaped Lift Distribution

$$\alpha_{eff} = \frac{[c_l \cdot t]_0 \cdot \sin^3 \Theta}{c_{l\alpha\infty}(\Theta) \cdot t(\Theta)} \qquad \text{(Equ. 00-04)}$$

The solution for the induced angle of attack is more complex. Truckenbrodt gives the induced angle of attack α_i for unswept wings:

$$\alpha_i = \frac{1}{2\pi} \cdot \int_{-1}^{+1} \frac{d\gamma}{dy'} \cdot \frac{dy'}{y - y'} \text{ with the circulation } \gamma = \frac{c_l \cdot t}{2b} \qquad \text{(Tru. 7.67 \& 7.70)}$$

Tru.

Using the following terms, we can transform this integrartion in the formulation as given in Multhopp:

$$y' = \cos\Theta \quad \frac{dy'}{d\Theta} = -\sin\Theta \quad \frac{d\gamma}{dy'} = \frac{d\gamma}{d\Theta} \cdot \frac{d\Theta}{dy'} = -\frac{d\gamma}{d\Theta} \cdot \frac{1}{\sin\Theta} \qquad \text{(Equ. 00-05)}$$

$$\alpha_i = \alpha_i(\Theta_p) = \frac{1}{2\pi} \cdot \int_0^{\pi} \frac{d\gamma}{d\Theta} \cdot \frac{d\Theta}{\cos\Theta - \cos\Theta_p} \qquad \text{(Mul. 13)}$$

Mul.

This formula now has to be solved for the bell shaped lift distribution:

$$\gamma(\Theta) = \gamma_0 \cdot \sin^3 \Theta \quad \frac{d\gamma}{d\Theta} = 3\gamma_0 \cdot \sin^2 \Theta \cdot \cos\Theta \qquad \text{(Equ. 00-06)}$$

$$\alpha_i(\Theta_p) = \frac{3\gamma_0}{2\pi} \cdot \int_0^{\pi} \frac{\sin^2 \Theta \cdot \cos\Theta}{\cos\Theta - \cos\Theta_p} d\Theta \qquad \text{(Equ. 00-07)}$$

For this the term is split up in several parts, which can be integrated seperately. The calculation is described in the **Appendix A1**. The final solution is:

$$\alpha_i(\Theta) = -\frac{3\gamma_0}{4}\cos(2\Theta) \text{ or } \alpha_i(\Theta) = -\frac{3[c_l \cdot t]_0}{8b}\cos(2\Theta) \qquad \text{(Equ. 00-08)}$$

This gives the required calculation method for the geometrical twist α_{geo}, which can be transformed, using equation (Equ. 00-03):

$$\alpha_{geo} = [c_l \cdot t]_0 \cdot \left\{ \frac{\sin^3 \Theta}{c_{l\alpha\infty}(\Theta) \cdot t(\Theta)} - \frac{3 \cdot \cos(2\Theta)}{8b} \right\} \qquad \text{(Equ. 00-09a)}$$

$$\alpha_{geo} = \frac{2 \cdot A \cdot c_L}{\pi \cdot b} \cdot \left\{ \frac{8 \cdot \sin^3 \Theta}{3 \cdot c_l(\Theta) \cdot t(\Theta)} - \frac{\cos(2\Theta)}{b} \right\} \qquad \text{(Equ. 00-09b)}$$

Appendix A1: Solution of Equation 00-07

$$\alpha_i\left(\Theta_p\right)=\frac{3\gamma_0}{2\pi}\cdot\int_0^\pi \frac{\sin^2\Theta\cdot\cos\Theta}{\cos\Theta-\cos\Theta_p}\,d\Theta \qquad \text{(Equ. 00-07)}$$

The integration is transformed into a set of terms, which can be solved seperately. An integratable solution was first presented by Dr. Martin MAURER[3], Munich, in 1995 and confirmed by the solution of Dr. Karl NICKEL, Freiburg i.Br., 1998. The solution of Dr. MAURER is basis for the here described calculation:

$$\frac{\sin^2\Theta\cdot\cos\Theta}{\cos\Theta-\cos\Theta_p}=\frac{\sin^2\Theta\cdot\cos\Theta+\left(\cos\Theta_p\sin^2\Theta-\cos\Theta_p\sin^2\Theta\right)}{\cos\Theta-\cos\Theta_p} \qquad \text{(Equ. A1-01a)}$$

$$\frac{\sin^2\Theta\cdot\cos\Theta}{\cos\Theta-\cos\Theta_p}=\frac{\sin^2\Theta\cdot\left[\cos\Theta-\cos\Theta_p\right]}{\cos\Theta-\cos\Theta_p}+\frac{\sin^2\Theta\cdot\cos\Theta_p}{\cos\Theta-\cos\Theta_p} \qquad \text{(Equ. A1-01b)}$$

$$\frac{\sin^2\Theta\cdot\cos\Theta}{\cos\Theta-\cos\Theta_p}=\sin^2\Theta+\cos\Theta_p\frac{\sin^2\Theta}{\cos\Theta-\cos\Theta_p} \qquad \text{(Equ. A1-01c)}$$

The following transformations can be used:

$$\sin^2\Theta=1-\cos^2\Theta \quad 2\cos^2\Theta-1=\cos(2\Theta) \quad 1=\cos(0\cdot\Theta) \qquad \text{(Equ. A1-02)}$$

$$\frac{\sin^2\Theta\cdot\cos\Theta}{\cos\Theta-\cos\Theta_p}=\sin^2\Theta+\frac{\cos\Theta_p}{2}\cdot\frac{\left(1-\cos(2\Theta)\right)}{\cos\Theta-\cos\Theta_p} \qquad \text{(Equ. A1-03a)}$$

$$\frac{\sin^2\Theta\cdot\cos\Theta}{\cos\Theta-\cos\Theta_p}=\sin^2\Theta+\frac{\cos\Theta_p}{2}\cdot\left\{\frac{\cos(0\cdot\Theta)}{\cos\Theta-\cos\Theta_p}-\frac{\cos(2\cdot\Theta)}{\cos\Theta-\cos\Theta_p}\right\} \qquad \text{(Equ. A1-03b)}$$

This leads to a new formula for the integration (Equ. 00-07):

$$\int_0^\pi\frac{\sin^2\Theta\cdot\cos\Theta}{\cos\Theta-\cos\Theta_p}d\Theta=\int_0^\pi\sin^2\Theta d\Theta+\frac{\cos\Theta_p}{2}\cdot\left\{\int_0^\pi\frac{\cos(0\cdot\Theta)}{\cos\Theta-\cos\Theta_p}d\Theta-\int_0^\pi\frac{\cos(2\cdot\Theta)}{\cos\Theta-\cos\Theta_p}d\Theta\right\}$$

(Equ. A1-04)

Solutions for the integrations are given in Multhopp, respectvely Bronstein[4]:

---

3 Persönliche Mitteilung an den Autor, Dr. M. MAURER, 1995

4 Taschenbuch der Mathematik, BRONSTEIN, SEMENDJAJEW, Deutsch, 20. Aufl., 1981

$$\int \sin^2 \Theta d\Theta = \left[\frac{\Theta}{2} - \frac{1}{4}\sin(2\Theta)\right] \qquad \text{(Bro. 275)}$$

Bro.

$$\int_0^\pi \frac{\cos(n\cdot\Theta)}{\cos\Theta - \cos\Theta_p} d\Theta = \pi \frac{\sin(n\cdot\Theta_p)}{\sin\Theta_p} \qquad \text{(Mul. 09)}$$

Mul.

Equation Equ. A1-04 now gives:

$$\int_0^\pi \frac{\sin^2\Theta\cdot\cos\Theta}{\cos\Theta - \cos\Theta_p} d\Theta = \left[\frac{\Theta}{2} - \frac{1}{4}\sin(2\Theta)\right]_0^\pi + \frac{\cos\Theta_p}{2}\cdot\left\{\pi\frac{\sin(0\cdot\Theta_p)}{\sin\Theta_p} - \pi\frac{\sin(2\cdot\Theta_p)}{\sin\Theta_p}\right\}$$

(Equ. A1-05)

and from this:

$$\sin 0 = 0 \quad \sin(2\Theta) = 2\sin\Theta\cos\Theta \quad 2\cos^2\Theta - 1 = \cos(2\Theta) \qquad \text{(Equ. A1 06)}$$

$$\int_0^\pi \frac{\sin^2\Theta\cdot\cos\Theta}{\cos\Theta - \cos\Theta_p} d\Theta = \left[\frac{\pi}{2}\right] + \frac{\cos\Theta_p}{2}\cdot\left\{0 - \pi\frac{2\cdot\sin\Theta_p\cos\Theta_p}{\sin\Theta_p}\right\} \qquad \text{(Equ. A1-07a)}$$

$$\int_0^\pi \frac{\sin^2\Theta\cdot\cos\Theta}{\cos\Theta - \cos\Theta_p} d\Theta = \frac{\pi}{2}\cdot\left\{1 - 2\cdot\cos^2\Theta_p\right\} \qquad \text{(Equ. A1-07b)}$$

$$\int_0^\pi \frac{\sin^2\Theta\cdot\cos\Theta}{\cos\Theta - \cos\Theta_p} d\Theta = -\frac{\pi}{2}\cdot\cos(2\Theta_p) \qquad \text{(Equ. A1-07c)}$$

This gives the required solution, which can be now written as equation Equ. 00-08:

$$\alpha_i(\Theta_p) = \frac{3\gamma_0}{2\pi}\cdot\int_0^\pi \frac{\sin^2\Theta\cdot\cos\Theta}{\cos\Theta - \cos\Theta_p} d\Theta \qquad \text{(Equ. 00-07)}$$

$$\alpha_i(\Theta) = -\frac{3\gamma_0}{4}\cos(2\Theta) \text{ or } \alpha_i(\Theta) = -\frac{3[c_l\cdot t]_0}{8b}\cos(2\Theta) \qquad \text{(Equ. 00-08)}$$

Appendix A2: Halfspan Center of Lift for the Bell-Shaped Distribution

The formula for total lift c_L is (Truckenbrodt):

Tru.

$$c_L = \frac{1}{A} \cdot \int_{-s}^{s} c_l \cdot t \cdot dx \qquad \text{(Tru. 5.42)}$$

For a symetrical lift distribution only the halfspan may be used:

$$c_L = \frac{s}{A} \cdot \int_{1}^{0} c_l \cdot t \cdot d\overline{x} \quad \text{with} \quad c_l \cdot t = [c_l \cdot t]_0 \cdot h(\overline{x}); \quad \overline{x} = x / s \qquad \text{(Equ. 00-01)}$$

If the spanwise control points are aranged along a "central-angle" Θ similar to the Multhopp-procedure, the following terms can be written:

Mul.

$$h(\overline{x}) = \sin^3 \Theta \ \text{und} \ \overline{x} = \cos\Theta \qquad \text{(Mul. 07)}$$

This gives the following solution for one side (halfspan):

$$c_L = \frac{s}{A} \cdot [c_l \cdot t]_0 \int_{0}^{\pi/2} \sin^4 \Theta \cdot d\Theta \ \text{mit} \ d\overline{x} = -\sin\Theta \cdot d\Theta \qquad \text{(Equ. 00-02)}$$

$$c_L = \frac{3\pi \cdot s}{16 \cdot A} \cdot [c_l \cdot t]_0 \qquad \text{(Equ. 00-03)}$$

The rolling-moment c_R of the halfwing is:

$$c_R = \frac{s}{A \cdot t_{ref}} \cdot \int_{1}^{0} c_l \cdot t \cdot x \cdot d\overline{x} \qquad \text{(Equ. A2-01)}$$

This gives now:

$$c_R = \frac{s}{A \cdot t_{ref}} \cdot [c_l \cdot t]_0 \int_{0}^{\pi/2} \sin^4 \Theta \cdot \cos\Theta \cdot d\Theta \qquad \text{(Equ. A2-02)}$$

Accordingly to Bronstein (Bro. 356) this gives:

$$c_R = \frac{s}{A \cdot t_{ref}} \cdot [c_l \cdot t]_0 \cdot \left[\frac{1}{5} \sin^5 \Theta\right]_0^{\pi/2} = \frac{s}{A \cdot t_{ref}} \cdot [c_l \cdot t]_0 \cdot \left[\frac{1}{5}\right] \qquad \text{(Equ. A2-03)}$$

The halfspan center of lift is then:

$$x_{Aero} = \frac{c_R \cdot t_{ref}}{c_L} = \frac{16}{15 \cdot \pi} = 0{,}33953 \qquad \text{(Equ. A2-04)}$$

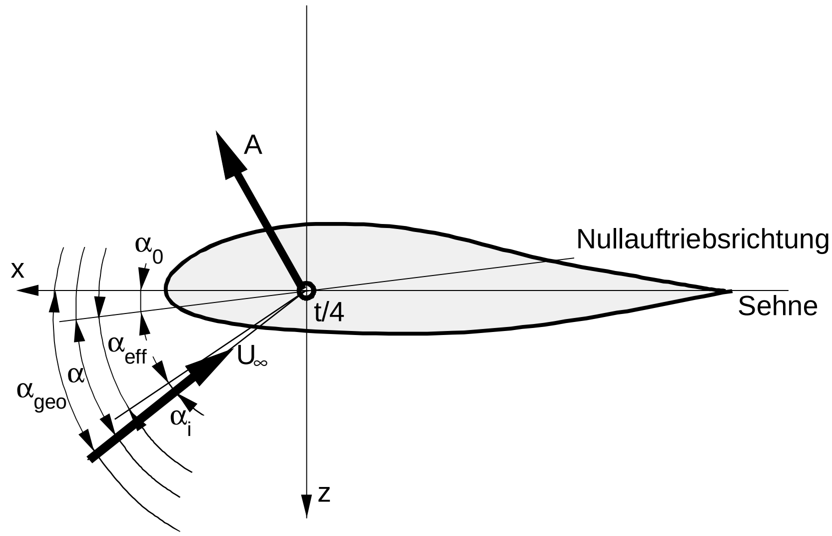
Appendix A3: Some Comments on the Bell-Shaped Lift Distribution

In many cases the discussion about the bell shaped lift distribution concentrates on the aerodynamic point of view. For the swept flying wing, this distribution has some advantages, but brings some drawbacks with respect to performance.
Unfortunately, one very important point is missed in all this discussion, the mathematics. Accordingly to reimar HORTEN, the bell shaped lift distribution was already used on the H II, long before Multhopp presented his calvculation method. An exact solution of the (unswept) lifting line was only available for the elliptical lift distribution ($c_a t \propto \sin^n \Theta$, n=1) at this time. For arbitrary wing layout only approximate methods as Schrenk or Lippisch were available. A solution for other functions $c_a t \propto \sin^n \Theta$ seem possible, but were not presented at this time. A solution of higher orders show that the lift distribution accordingly the power n=2 is physically not possible. It requires an indefinite twist at the wing tip. Next possibility is the power n=3, and trhis is the bell shaped lift distribution. The required wing twist includes the induced angle of attack which follows a simple function $\alpha_i \propto \cos(2\Theta)$. Twist at the tip is the (negative) same on as at the root.
The reconstruction of this solution was tricky but for a mathematician like R. Horten possible. It enables a fast method to calculate the required basic twist, making the layout independent from a method like Multhiopp´s. Unfortunately we miss the final verification that Reimar Horten has used this simple method. The surviving calculations give no description how the twist was derived. Nevertheless, the twist fits well with the bell shape, at least for some of the airplanes.
Stress calculations then were performed using the well established Lippisch method[5].

rev. 01: 98-01-05
rev. 03: 99-01-02

5 A. Lippisch: Bestimmung der Auftriebsverteilung längs der Spannweite, Flugsport 26, 1933 und ff.

Appendix A4: Geometry to the Bell-Shaped Lift Distribution



Sehne: Wing Section Axle
Nullauftriebsrichtung: Direction for Zero Lift Flow